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Integrability of One-Parameter Groups Generated by the Virasoro Operators

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Local one-parameter groups generated by the Virasoro operators were constructed via Feynman path integrals on a coadjoint of the infinite-dimensional Heisenberg group in the previous paper [T. Hashimoto, 1996. *J. Funct. Anal.* **137**, 191–218]. The main purpose of this paper is to prove that the one-parameter groups satisfy the integrability condition, i.e., the actions of the one-parameter subgroups are compatible with the action of the Virasoro algebra. © 1997 Academic Press

0. INTRODUCTION

The Virasoro algebra, which we denote by Vir in this paper, is a central extension of $\mathrm{Vect}_{\mathbb{C}}(S^1)$, the Lie algebra consisting of $\mathbb{C}$-valued polynomial vector fields on the unit circle. It has a basis $\{d_p = ie^{ip\theta}(d/d\theta),\ p \in \mathbb{Z};\ c\}$:

$$\mathrm{Vir} = \sum_{p \in \mathbb{Z}} \mathbb{C} d_p \oplus \mathbb{C} c.$$

The commutation relations among the basis are given by

$$[d_p, d_q] = (p-q)\, d_{p+q} + \frac{p^3 - p}{12}\, \delta_{p+q,0}\, c, \qquad [d_p, c] = 0.$$

It is well known that one can construct a representation of Vir, say π, on the $\mathbb{C}$-vector space $\mathscr{V}$ of polynomials in infinitely many variables, called the Fock space, i.e., a representation of the infinite-dimensional Heisenberg algebra (see, e.g., [KR].) The operator $L_p = \pi(d_p)$ on the Fock space $\mathscr{V}$ is called the Virasoro operator.

In [FK], Frenkel and Kac gave the explicit formulas of one-parameter groups generated by L_p for $p = 0, \pm 1$. They realized a complex polarization of the infinite-dimensional Heisenberg algebra as a Hardy space. The one-parameter groups act on the space of holomorphic functionals on the

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Hardy space. On the other hand, Goodman and Wallach showed in [GW] that every unitary projective representation of $\mathrm{Vect}^{\infty}(S^1)$, the Lie algebra consisting of $\mathbb{R}$-valued smooth vector fields on S^1, lifts to a projective unitary representation of $\mathrm{Diff}(S^1)$, the group of orientation-preserving diffeomorphisms of S^1.

In [H], the author constructed local one-parameter groups generated by L_p for $p \in \mathbb{Z}$ via Feynman path integrals on an coadjoint orbit of the infinite-dimensional Heisenberg group, along the same line as in [HO²SY, HO²S1,2], in the case where L_0 acts trivially on $\mathscr{V}$. He gave explicit formulas of the action of the local one-parameter groups on the algebra $\mathscr{U}$ of exponential functions, realizing the infinite-dimensional Heisenberg group as a central extension of the loop group consisting of all smooth mappings of S^1 into $\mathbb{C}$ such that $\int_0^{2\pi} f(\theta)\, d\theta = 0$, and taking, as a complex polarization, the closed subspace consisting of the mappings which can be extended to holomorphic functions on the unit disk vanishing at the origin. Notice that $\mathscr{U}$ is dense in $\mathscr{H}$, where $\mathscr{H}$ is the Hilbert space completion of $\mathscr{V}$ with respect to an inner product $\langle \cdot | \cdot \rangle$ (see (1.3) below). If $p > 0$, then the local one parameter groups are in fact one-parameter groups. However, if we construct the Heisenberg group from *polynomial* loops, we shall obtain one-parameter groups generated by L_p, which we denote by e^{zL_p} for $z \in \mathbb{C}$ and $p \in \mathbb{Z}$. We employ the polynomial realization of the Heisenberg group in this paper.

The purpose of this paper is to prove that the one-parameter groups e^{zL_p} generated by the Virasoro operators L_p satisfy the commutation relation

$$\left.\frac{\partial^2}{\partial z\, \partial w}\right|_{z=w=0} e^{zL_p} e^{wL_q} e^{-zL_p} e^{-wL_q} = [L_p, L_q] \tag{0.1}$$

and the integrability condition

$$e^{-zL_p} \pi(X)\, e^{zL_p} = \exp(-z \operatorname{ad} L_p)\, \pi(X) \tag{0.2}$$

for $z, w \in \mathbb{C}$, $p, q \in \mathbb{Z}$ and $X \in \mathrm{Vir}$. The key step to prove (0.2) can be reduced to a special case of the Gauss formula,

$$F(a, b; c; 1) = \frac{\Gamma(c)\, \Gamma(c-a-b)}{\Gamma(c-a)\, \Gamma(c-b)} \qquad (\operatorname{Re} c > 0,\ \operatorname{Re}(c-a-b) > 0),$$

with a nonpositive integer, where F and Γ denote the hypergeometric function and the gamma function, respectively (see [W]).

Here we have to give a precise meaning to the composition of operators, since the image of e^{zL_p} is not contained in $\mathscr{U}$, but in $\mathscr{H}$ if $p < 0$. The fact that a linear operator $A: \mathscr{U} \to \mathscr{H}$ is in one-to-one correspondence with matrix elements $\langle u|\, A\, |v\rangle$ under the correspondence

$$\langle u|\, A\, |v\rangle = \langle u\,|\, Av\rangle \qquad (u, v \in \mathscr{U})$$

enables us to define a composition of operators by matrix elements (see [TK]). We shall show that the identities (0.1) and (0.2) hold in this sense.

This paper is organized as follows. In Section 1, we introduce some notations, give one-parameter groups generated by the Virasoro operators which we can obtain by calculating path integrals on a coadjoint orbit of the infinite dimensional Heisenberg group, as mentioned above, and define a composition of operators by matrix elements following [TK]. Section 2 is devoted to a proof of the commutation relations (0.1) among the one-parameter groups. Section 3 is devoted to a proof of the integrability condition (0.2) of the one-parameter groups.

Finally, we shall follow the convention that any sum $\sum_{k=p}^{q}$ should vanish if $p > q$ throughout the paper.

1. VIRASORO OPERATORS

Let E be a direct sum of infinitely many copies of $\mathbb{C}$. We write an element of E as $\mathbf{x} = (x_n)_{n=1}^{\infty}$. Notice that if $\mathbf{x} \in E$ then $x_n = 0$ for n sufficiently large. We define a bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ on E by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{n \geqslant 1} n x_n y_n$$

and a conjugation by $\bar{\mathbf{x}} = (\bar{x}_n)_{n=1}^{\infty}$ for $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n) \in E$.

The Virasoro algebra Vir has a basis $\{d_n (n \in \mathbb{Z}), c\}$:

$$\mathrm{Vir} = \sum_{n \in \mathbb{Z}} \mathbb{C} d_n \oplus \mathbb{C} c.$$

The commutation relations among the basis are given by

$$[d_n, d_m] = (n-m)\, d_{n+m} + \frac{n^3 - n}{12}\, \delta_{n+m,0}\, c,$$

$$[d_n, c] = 0.$$

It is well known that one can construct representations of Vir through representations of the infinite-dimensional Heisenberg algebra $\mathfrak{g}$, i.e., the Lie algebra with a basis $\{a_n (n \in \mathbb{Z} \backslash \{0\}), K\}$:

$$\mathfrak{g} = \sum_{n \neq 0} \mathbb{C} a_n \oplus \mathbb{C} K.$$

The commutation relations among the basis are

$$[a_n, a_m] = m \delta_{n+m, 0} K, \qquad [a_n, K] = 0.$$

Let $\mathscr{V} = \mathbb{C}[x_1, x_2, ...]$, the algebra of polynomials in infinitely many variables $x_1, x_2, ...$ over $\mathbb{C}$. Then we define a representation π of $\mathfrak{g}$ on $\mathscr{V}$ by

$$\pi(a_n) = -\frac{\partial}{\partial x_n}, \qquad \pi(a_{-n}) = n x_n, \qquad \pi(K) = 1$$

for $n = 1, 2, ...$ Put $a(n) = \pi(a_n)$ for $n \in \mathbb{Z} \backslash \{0\}$ and $a(0) = 0$ (zero operator) for brevity.

For an integer p, the Virasoro operator L_p, which acts on $\mathscr{V}$, is defined by

$$L_p = -\tfrac{1}{2} \sum_{n \in \mathbb{Z}} {}^{\circ}_{\circ} a(-n)\, a(n+p) {}^{\circ}_{\circ},$$

where ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ denotes the normal ordering given by

$${}^{\circ}_{\circ} a(n)\, a(m) {}^{\circ}_{\circ} = \begin{cases} a(n)\, a(m) & (n \leqslant m) \\ a(m)\, a(n) & (n \geqslant m). \end{cases}$$

Then the operators $\{L_p\}_{p \in \mathbb{Z}}$ satisfy the commutation relations

$$[L_p, L_q] = (p-q)\, L_{p+q} + \frac{p^3 - p}{12} \delta_{p+q, 0}. \tag{1.1}$$

Namely, Vir acts on $\mathscr{V}$ by the assignment

$$d_p \mapsto L_p, \qquad c \mapsto 1. \tag{1.2}$$

We also denote the action (1.2) of Vir by $(\pi, \mathscr{V})$.

Let us introduce an inner product $\langle \cdot \mid \cdot \rangle$ on $\mathscr{V}$, which is linear in the first argument and anti-linear in the second, by defining

$$\langle x_1^{i_1} x_2^{i_2} x_3^{i_3} \cdots \mid x_1^{j_1} x_2^{j_2} x_3^{j_3} \cdots \rangle = \prod_{k=1}^{\infty} \left(\frac{i_k!}{k^{i_k}} \right) \delta_{i_k, j_k} \tag{1.3}$$

for monomials $x_1^{i_1}x_2^{i_2}x_3^{i_3}\cdots$ and $x_1^{j_1}x_2^{j_2}x_3^{j_3}\cdots$ of $\mathscr{V}$. We denote by $\mathscr{H}$ the Hilbert space completion of $\mathscr{V}$ with respect to the inner product.

Remark. The inner product $\langle\cdot|\cdot\rangle$ defined in (1.3) can be written as

$$\langle u|v\rangle=\int_E\prod_{n=1}^{\infty}\frac{i}{2\pi}n\,dx_n\,d\bar{x}_n\,e^{-n|x_n|^2}u(\mathbf{x})\,\overline{v(\mathbf{x})} \tag{1.4}$$

if we regard $u, v\in\mathscr{V}$ as functions on E. Notice that since $u, v\in\mathscr{V}$ depend only on finite number of the x_n, the integral reduces to the one on a finite-dimensional space $\mathbb{C}^N$ for some N. We write the integral in (1.4) as $\int d\mu(\mathbf{x})\,u\bar{v}$ for brevity.

Let $\mathscr{U}$ be the algebra generated by exponential functions on E, i.e., elements of the form $\exp\langle\mathbf{c},\mathbf{x}\rangle\in\mathscr{H}$ with $\mathbf{c}\in E$. Then $\mathscr{U}$ is dense in $\mathscr{H}$. It follows immediately from (1.4) that

$$\langle\exp\langle\mathbf{b},\mathbf{x}\rangle\,|\exp\langle\mathbf{c},\mathbf{x}\rangle\rangle=\exp\langle\mathbf{b},\bar{\mathbf{c}}\rangle$$

for $\mathbf{b},\mathbf{c}\in E$, and therefore, that $\int d\mu(\mathbf{z})\,|e_{\mathbf{z}}\rangle\langle e_{\mathbf{z}}|$ is the identity operator on $\mathscr{U}$,

$$\int d\mu(\mathbf{z})\langle u\mid e_{\mathbf{z}}\rangle\langle e_{\mathbf{z}}|=u,\qquad\int d\mu(\mathbf{z})\,|e_{\mathbf{z}}\rangle\langle e_{\mathbf{z}}|u\rangle=u \tag{1.5}$$

for any $u\in\mathscr{U}$, where $e_{\mathbf{z}}$ denotes the element $\exp\langle\mathbf{z},\mathbf{x}\rangle\in\mathscr{U}$.

Definition. Let p be a nonnegative integer and z a complex number. For $\mathbf{c}=(c_n)_{n=1}^{\infty}\in E$, we put

$$D_p(\mathbf{c},z)_n=\sum_{k\geqslant 0}\frac{z^k}{k!}(n+p)\cdots(n+kp)\,c_{n+kp} \tag{1.6}$$

and $D_p(\mathbf{c},z)=(D_p(\mathbf{c},z)_n)_{n=1}^{\infty}\in E$. Furthermore, we put

$$\begin{aligned}\eta_p(\mathbf{c};z)=\sum_{n=1}^{p-1}\sum_{k,l\geqslant 0}&\frac{z}{k+l+1}\frac{z^k}{k!}n(n+p)\cdots(n+kp)\,c_{n+kp}\\&\times\frac{z^l}{l!}(p-n)(p-n+p)\cdots(p-n+lp)\,c_{p-n+lp}.\end{aligned} \tag{1.7}$$

We call η_p *a quadratic term* in this paper.

Now we define linear operators e^{zL_p} and $e^{zL_{-p}}$ of $\mathscr{U}$ into $\mathscr{H}$ by putting

$$\begin{aligned} e^{zL_p}u &= \exp(-\tfrac{1}{2}\eta_p(\mathbf{c}; z) + \langle D_p(\mathbf{c}, z), \mathbf{x}\rangle), \\ e^{zL_{-p}}u &= \exp(-\tfrac{1}{2}\eta_p(\mathbf{x}; z) + \langle D_p(\mathbf{x}, z), \mathbf{c}\rangle), \end{aligned} \tag{1.8}$$

for $u = \exp\langle \mathbf{c}, \mathbf{x}\rangle$ and extending them linearly.

Notice that if $p \geqslant 0$ then

$$D_p(\mathbf{c}, z+w) = D_p(D_p(\mathbf{c}, z), w), \tag{1.9}$$

$$\eta_p(\mathbf{c}; z+w) = \eta_p(\mathbf{c}; z) + \eta_p(D_p(\mathbf{c}, z); w) \tag{1.10}$$

for all $\mathbf{c} \in E$ and $z, w \in \mathbb{C}$, and the image of $\mathscr{U}$ by e^{zL_p} is contained in $\mathscr{U}$. Thus, it follows immediately from (1.9) and (1.10) that if $p \geqslant 0$ then the operators e^{zL_p} enjoy the semigroup property

$$e^{zL_p}e^{wL_p} = e^{(z+w)L_p} \tag{1.11}$$

for $z, w \in \mathbb{C}$. (cf. Theorem 3.1 in [H].)

In order to show that (1.11) holds if $p < 0$, we need to define a composition of operators of $\mathscr{U}$ into $\mathscr{H}$.

For a linear operator $A\colon \mathscr{U} \to \mathscr{H}$, we can define a sesquilinear form $\hat{A}\colon \mathscr{U} \times \mathscr{U} \to \mathbb{C}$ by

$$\hat{A}(u, v) = \langle u | Av\rangle. \tag{1.12}$$

Then $\hat{A}(\cdot, v)$ with $v \in \mathscr{U}$ fixed is a continuous linear form on $\mathscr{U}$. Conversely, for a sesquilinear form $\hat{A}$ on $\mathscr{U} \times \mathscr{U}$ such that $\hat{A}(\cdot, v)$ is continuous on $\mathscr{U}$ with $v \in \mathscr{U}$ fixed, there exits a linear operator $A\colon \mathscr{U} \to \mathscr{H}$ such that (1.12) holds. Namely, a linear operator $\mathscr{U} \to \mathscr{H}$ is in one-to-one correspondence with such a sesquilinear form on $\mathscr{U} \times \mathscr{U}$. For a linear operator $A\colon \mathscr{U} \to \mathscr{H}$, $\hat{A}(u, v)$ is called *a matrix element*, which we denote by $\langle u|\, A\, |v\rangle$. Thus, giving a linear operator is equivalent to giving matrix elements (see [TK]).

Definition. Let $\{A_1, A_2, ..., A_N\}$ be a sequence of linear operators of $\mathscr{U}$ into $\mathscr{H}$. Then we define a composition $A_1 A_2 \cdots A_N$ by giving its matrix elements,

$$\begin{aligned} &\langle u|\, A_1 A_2 \cdots A_n\, |v\rangle \\ &\quad = \int \cdots \int d\mu(\mathbf{z}_1)\, d\mu(\mathbf{z}_2) \cdots d\mu(\mathbf{z}_{N-1}) \\ &\qquad \times \langle u|\, A_1\, |e_{\mathbf{z}_1}\rangle\langle e_{\mathbf{z}_1}|\, A_2\, |e_{\mathbf{z}_2}\rangle\langle e_{\mathbf{z}_2}| \cdots |e_{\mathbf{z}_{N-1}}\rangle\langle e_{\mathbf{z}_{N-1}}|\, A_N\, |v\rangle \end{aligned}$$

with $u, v \in \mathscr{U}$.

Remark. If the images of A_i are contained in $\mathscr{U}$ for $i=1, 2, ..., N-1$ and if the domains of the adjoint of A_i for $i=2, 3, ..., N$ contain $\mathscr{U}$, then, using (1.5), we can show

$$\langle u|\, A_1 A_2 \cdots A_N\, |v\rangle = \langle u|\, A_1 A_2 \cdots A_N\, |v\rangle$$

for all $u, v \in \mathscr{U}$.

We can prove the following proposition in the same way as Theorem 3.4 in [H].

PROPOSITION 1.1. *Let p be a nonnegative integer and z a complex number. For any elements u and v of $\mathscr{U}$, we have*

$$\langle e^{zL_p} u \,|\, v\rangle = \langle u \,|\, e^{\bar{z}L_{-p}} v\rangle.$$

Then, it is obvious by (1.11) and Proposition 1.1 that (1.11) also holds for $p<0$, where the composition is in the sense above (see Corollary 3.5 in [H].)

For $p \in \mathbb{Z}$ and $\mathbf{c} = (c_n)_{n=1}^{\infty} \in E$, let $d_p(\mathbf{c})$ be an element of E whose nth components are given by $(n+p)\, c_{n+p} \cdot \theta(n+p)$, where $\theta(x)$ is the Heaviside function, i.e., $\theta(x)=0$ if $x \leqslant 0$ and $\theta(x)=1$ if $x>0$. Notice that

$$\left.\frac{d}{dz}\right|_{z=0} D_p(\mathbf{c}, z) = d_p(\mathbf{c}), \qquad \langle d_p(\mathbf{b}), \mathbf{c}\rangle = \langle \mathbf{b}, d_{-p}(\mathbf{c})\rangle$$

for $\mathbf{b}, \mathbf{c} \in E$, and that

$$L_p u = \left(-\tfrac{1}{2} \sum_{n=1}^{p-1} n(p-n)\, c_n c_{p-n} + \langle d_p(\mathbf{c}), \mathbf{x}\rangle \right) u, \tag{1.12}$$

$$L_{-p} u = \left(-\tfrac{1}{2} \sum_{n=1}^{p-1} n(p-n)\, x_n x_{p-n} + \langle d_{-p}(\mathbf{c}), \mathbf{x}\rangle \right) u \tag{1.13}$$

for $u = \exp \langle \mathbf{c}, \mathbf{x}\rangle \in \mathscr{U}$, if $p \geqslant 0$.

Now, for $p \in \mathbb{Z}$ and $z \in \mathbb{C}$, we extend the domain of e^{zL_p} from $\mathscr{U}$ to $\mathscr{V} \otimes \mathscr{U}$. Since an element $f(x_1, x_2, x_3, ...)\, e^{\langle \mathbf{c}, \mathbf{x}\rangle} \in \mathscr{V} \otimes \mathscr{U}$ can be written as

$$f(x_1, x_2, x_3, ...)\, e^{\langle \mathbf{c}, \mathbf{x}\rangle} = f(D_1, D_2, D_3, ...)\, e^{\langle \mathbf{c}, \mathbf{x}\rangle}, \tag{1.14}$$

where $f \in \mathscr{V}$ and $D_n = (1/n)(\partial/\partial c_n)$, we define the action of e^{zL_p} on the element by

$$e^{zL_p}(f(x_1, x_2, x_3, ...)\, e^{\langle \mathbf{c}, \mathbf{x}\rangle}) = f(D_1, D_2, D_3, ...)\, e^{zL_p}(e^{\langle \mathbf{c}, \mathbf{x}\rangle}).$$

Then it follows from definition and Proposition 1.1 that

$$\langle e^{zL_p} u \mid v \rangle = \langle u \mid e^{\bar{z} L_{-p}} v \rangle$$

for all $u, v \in \mathscr{V} \otimes \mathscr{U}$.

Finally, let us introduce *a polarized quadratic term.* For $\mathbf{b} = (b_n)$, $\mathbf{c} = (c_n) \in E$ and $z \in \mathbb{C}$, we put

$$\eta_p(\mathbf{b}, \mathbf{c}; z) = \sum_{n=1}^{p-1} \sum_{k,\, l \geqslant 0} \frac{z}{k+l+1} \frac{z^k}{k!} n(n+p) \cdots (n+kp)\, b_{n+kp}$$
$$\times \frac{z^l}{l!} (p-n)(p-n+p) \cdots (p-n+lp)\, c_{p-n+lp}. \tag{1.15}$$

Then, of course, we have

$$\eta_p(\mathbf{b}; z) = \eta_p(\mathbf{b}, \mathbf{b}; z).$$

2. COMMUTATION RELATION

This section is devoted to a proof of the commutation relations among the one-parameter groups e^{zL_p} given is Section 1. Throughout the section, we assume the parameters s and t to be real for brevity.

PROPOSITION 2.1. *Let p and q be nonnegative integers. Then for any element u of $\mathscr{U}$, we have*

$$\left. \frac{\partial^2}{\partial s\, \partial t} \right|_{s=t=0} e^{sL_p} e^{tL_q} e^{-sL_p} e^{-tL_q} u = [L_p, L_q]\, u. \tag{2.1}$$

Proof. We may assume that u is of the form $\exp \langle \mathbf{c}, \mathbf{x} \rangle$. Defining $\varphi(s, t)$ by

$$e^{\varphi(s,t)} = e^{sL_p} e^{tL_q} e^{-sL_p} e^{-tL_q} u,$$

we obtain from (1.18) that

$$\varphi(s, t) = -\tfrac{1}{2} \eta_q(\mathbf{c}; -t) - \tfrac{1}{2} \eta_p(D_q(\mathbf{c}, -t); -s) - \tfrac{1}{2} \eta_q(D_p(D_q(\mathbf{c}, -t), -s); t)$$
$$- \tfrac{1}{2} \eta_q(D_q(D_p(D_q(\mathbf{c}, -t), -s), t); s)$$
$$+ \langle D_p(D_q(D_p(D_q(\mathbf{c}, -t), -s), t), s), \mathbf{x} \rangle. \tag{2.2}$$

First we consider the last term in (2.2). Since we have

$$\begin{aligned}
&D_p(D_q(\mathbf{c}, t), s)_n \\
&\quad = \sum_{k \geqslant 0} \frac{s^k}{k!} (n+p) \cdots (n+kp) \sum_{l \geqslant 0} \frac{t^l}{l!} (n+kp+q) \cdots (n+kp+lq)\, c_{n+kp+lq} \\
&\quad = c_n + s(n+p)\, c_{n+p} + t(n+q)\, c_{n+q} + st(n+p)(n+p+q)\, c_{n+p+q} \\
&\qquad + O(s^2, t^2)
\end{aligned}$$

by (1.6), it follows that

$$\begin{aligned}
&\langle D_p(D_q(D_p(D_q(\mathbf{c}, -t), -s), t), s), \mathbf{x} \rangle \\
&\quad = \sum_{n \geqslant 1} n x_n (D_p(D_q(\mathbf{c}, -t), -s)_n + s(n+p)\, D_p(D_q(\mathbf{c}, -t), -s)_{n+p} \\
&\qquad + t(n+q)\, D_p(D_q(\mathbf{c}, -t), -s)_{n+q} \\
&\qquad + st(n+p)(n+p+q)\, D_p(D_q(\mathbf{c}, -t), -s)_{n+p+q}) \\
&\qquad + O(s^2, t^2) \\
&\quad = \sum_{n \geqslant 1} n x_n (c_n + st(p-q)(n+p+q)\, c_{n+p+q}) + O(s^2, t^2).
\end{aligned} \tag{2.3}$$

Now we turn to the quadratic terms in (2.2). Since we have

$$\eta_p(\mathbf{c}; t) = t \sum_{n=1}^{p-1} n(p-n)\, c_n c_{p-n} + O(t^2)$$

by (1.7), it follows that

$$\begin{aligned}
\eta_p(D_q(\mathbf{c}, -t); -s) &= -s \sum_{n=1}^{p-1} n(p-n)\, D_q(\mathbf{c}, -t)_n\, D_q(\mathbf{c}, -t)_{p-n} \\
&= -s \sum_{n=1}^{p-1} n(p-n)(c_n c_{p-n} - t((n+q)\, c_{n+q} c_{p-n} \\
&\qquad + (p-n+q)\, c_n c_{p-n+q})) + O(s^2, t^2).
\end{aligned} \tag{2.4}$$

Similarly, we obtain

$$\begin{aligned}
\eta_q(D_p(D_q(\mathbf{c}, -t), -s); t) &= t \sum_{n=1}^{q-1} n(q-n)(c_n c_{q-n} - s((n+p)\, c_{n+p} c_{q-n} \\
&\qquad + (q-n+p)\, c_n c_{q-n+p})) + O(s^2, t^2),
\end{aligned}$$

$$\eta_p(D_q(D_p(D_q(\mathbf{c}, -t), -s), t); s) = s \sum_{n=1}^{p-1} n(p-n)\, c_n c_{p-n} + O(s^2, t^2).$$

Thus, the sum of the quadratic terms in $\varphi(s, t)$ equals

$$\begin{aligned} -\tfrac{1}{2} \sum_{n=1}^{p-1} n(p-n)((n+q)\, c_{n+q} c_{p-n} + (p-n+q)\, c_n c_{p-n+q}) \\ + \tfrac{1}{2} \sum_{n=1}^{q-1} n(q-n)((n+p)\, c_{n+p} c_{q-n} + (q-n+p)\, c_n c_{q-n+p}) \\ = -\tfrac{1}{2} \sum_{n=1}^{p+q-1} (p-q)\, n(p+q-n)\, c_n c_{p+q-n} + O(s^2, t^2). \end{aligned} \tag{2.5}$$

Therefore it follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned} \left.\frac{\partial^2}{\partial s\, \partial t}\right|_{s=t=0} e^{\varphi(s, t)} = (p-q) \Bigg(\sum_{n \geqslant 1} n x_n (n+p+q)\, c_{n+p+q} \\ - \frac{1}{2} \sum_{n=1}^{p+q-1} n(p+q-n)\, c_n c_{p+q-n} \Bigg), \end{aligned}$$

which equals the right-hand side of (2.1) by (1.12). ∎

PROPOSITION 2.2. *Let p and q be nonnegative integers. Then for any elements u and v of $\mathcal{U}$, we have*

$$\left.\frac{\partial^2}{\partial s\, \partial t}\right|_{s=t=0} \langle u|\, e^{sL_{-p}} e^{tL_q} e^{-sL_{-p}} e^{-tL_q}\, |v\rangle = \langle u|\, [L_{-p}, L_q]\, |v\rangle. \tag{2.6}$$

Proof. Using (1.5) and Proposition 1.1, we can show that

$$\langle u|\, e^{sL_{-p}} e^{tL_q} e^{-sL_{-p}} e^{-tL_q}\, |v\rangle = \langle e^{tL_{-q}} e^{sL_p} u \,|\, e^{-sL_{-p}} e^{t-L_q} v\rangle.$$

We may assume that $u = \exp\langle \mathbf{b}, \mathbf{x}\rangle$ and $v = \exp\langle \mathbf{c}, \mathbf{x}\rangle$ with $\mathbf{b}, \mathbf{c} \in E$, as usual. Setting $e^{tL_{-q}} e^{sL_p} u = \exp \phi(s, t)$ and $e^{-sL_{-p}} e^{-tL_q} v = \exp \psi(s, t)$, we obtain from (1.8) that

$$\begin{aligned} \phi(s, t) = -\frac{s}{2} \sum_{n=1}^{p-1} n(p-n)\, b_n b_{p-n} - \frac{t}{2} \sum_{n=1}^{q-1} n(q-n)\, x_n x_{q-n} \\ + \sum_{n \geqslant 1} n b_n x_n + sn(n+p)\, b_{n+p} x_n + tn(n+q)\, b_n x_{n+q} \\ + st\, n(n+p)(n+q)\, b_{n+p} x_{n+q} + O(s^2, t^2), \end{aligned}$$

$$\psi(s,t)=\frac{s}{2}\sum_{n=1}^{p-1} n(p-n)\,x_n x_{p-n}+\frac{t}{2}\sum_{n=1}^{q-1} n(q-n)\,c_n c_{q-n}$$
$$+\sum_{n\geqslant 1} nc_n x_n - sn(n+p)\,c_n x_{n+p}-tn(n+q)\,c_{n+q}x_n$$
$$+st\,n(n+p)(n+q)\,c_{n+q}x_{n+p}+O(s^2,t^2).$$

Since

$$x_1^{j_1}x_2^{j_2}\cdots x_k^{j_k}e^{\langle \mathbf{b},\,\mathbf{x}\rangle}=\left(\frac{\partial}{\partial b_1}\right)^{j_1}\left(\frac{1}{2}\frac{\partial}{\partial b_2}\right)^{j_1}\cdots\left(\frac{1}{k}\frac{\partial}{\partial b_k}\right)^{j_k}e^{\langle \mathbf{b},\,\mathbf{x}\rangle}$$

for all $\mathbf{b}=(b_n)_{n=1}^{\infty}\in E$ (cf. (1.14)), simple but lengthy calculation shows that

$$\frac{\partial^2}{\partial s\,\partial t}\bigg|_{s=t=0}\langle \exp\phi(s,t)\,|\,\exp\psi(s,t)\rangle$$
$$=\Bigg(-\frac{p^3-p}{12}\,\delta_{p,q}-\sum_{n=1}^{p-1}\sum_{m=1}^{q-1} m(q-m)(p-n)\,b_{p-n}\bar{c}_{q-m}\delta_{n,m}$$
$$+\sum_{n\geqslant 1} n(n+p)(n+q)\,b_{n+p}\bar{c}_{n+q}-\sum_{n,\,m\geqslant 1} nm(m+q)\,b_m\bar{c}_n\delta_{n+p,\,m+q}$$
$$+\sum_{n=1}^{q-1}\sum_{m\geqslant 1} mn(q-n)\,\bar{c}_m\bar{c}_{q-n}\delta_{n,\,m+p}$$
$$+\sum_{m\geqslant 1}\sum_{n=1}^{p-1} mn(p-n)\,b_m b_{p-n}\delta_{n,\,m+q}\Bigg)e^{\langle \mathbf{b},\,\bar{\mathbf{c}}\rangle}. \tag{2.7}$$

It is easy to show that (2.7) equals

$$-\frac{p^3-p}{12}\,\delta_{p,q}e^{\langle \mathbf{b},\,\bar{\mathbf{c}}\rangle}-(p+q)\Bigg(-\frac{1}{2}\sum_{n=1}^{p-q-1} n(p-q-n)\,b_n b_{p-q-n}$$
$$+\sum_{n\geqslant 1} n(n+p-q)\,b_{p-q+n}\bar{c}_n\Bigg)e^{\langle \mathbf{b},\,\bar{\mathbf{c}}\rangle},$$

if $p\geqslant q$, and equals

$$-(p+q)\Bigg(-\tfrac{1}{2}\sum_{n=1}^{q-p-1} n(q-p-n)\,\bar{c}_n\bar{c}_{q-p-n}$$
$$+\sum_{n\geqslant 1} n(n+q-p)\,b_n\bar{c}_{q-p-n}\Bigg)e^{\langle \mathbf{b},\,\bar{\mathbf{c}}\rangle},$$

if $p<q$, which, in both cases, coincides with the right-hand side of (2.6). This completes the proof. ∎

THEOREM 2.3. *Let p, q be any integers. Then we have*

$$\left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s=t=0} e^{sL_p}e^{tL_q}e^{-sL_p}e^{-tL_q}=[L_p, L_q].$$

Proof. By the definition of the composition of the operators of $\mathscr{U}$ of $\mathscr{H}$, it is enough to show that

$$\left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s=t=0} \langle u|\, e^{sL_p}e^{tL_q}e^{-sL_p}e^{-tL_q}\,|v\rangle=\langle u|\,[L_p, L_q]\,|v\rangle \tag{2.8}$$

for any elements u and v of $\mathscr{U}$.

We have already seen that (2.8) holds if $q\geqslant 0$. So, let us assume that $q<0$. By (1.5) and Proposition 1.1, we obtain that

$$\langle u|\, e^{sL_p}e^{tL_q}e^{-sL_p}e^{-tL_q}\,|v\rangle=\overline{\langle v|\, e^{-tL_{-q}}e^{-sL_{-p}}e^{tL_{-q}}e^{sL_{-p}}\,|u\rangle}$$

for $u, v\in\mathscr{U}$. Therefore, it follows from Propositions 2.1 and 2.2 that (2.8) also holds if $q<0$. This completes the proof of the theorem. ∎

3. INTEGRABILITY

In this section we prove that the one-parameter groups e^{zL_p} given in Section 1 satisfy the integrability condition. For $a\in\mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$ and $n\in\mathbb{Z}_{\geqslant 0}$, we denote $\Gamma(a+n)/\Gamma(a)$ by $(a)_n$, where Γ denotes the gamma function. Notice that if $n\in\mathbb{Z}$ then $(a)_n$ is well defined and satisfies

$$(1-a)_{-n}=\frac{(-)^n}{(a)_n} \tag{3.1}$$

for $a\in\mathbb{C}\backslash\mathbb{Z}$.

We prepare two easy lemmas. Though the first one is a special case of the Gauss formula (see Remark 1 below), we give its direct and easy proof.

LEMMA 3.1. *Let ν be a nonnegative integer. For $\alpha, \beta\in\mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$ such that $\beta-\alpha\in\mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$, we have*

$$\sum_{j=0}^{\nu}\binom{\nu}{j}(-)^j\frac{(\alpha)_j}{(\beta)_j}=\frac{(\beta-\alpha)_\nu}{(\beta)_\nu}. \tag{3.2}$$

Proof. We prove (3.2) by induction on ν. It is trivial for $\nu=0$. Assume that (3.2) is true for $\nu \geqslant 0$. Then

$$\begin{aligned}\sum_{j=0}^{\nu+1}\binom{\nu+1}{j}(-)^j\frac{(\alpha)_j}{(\beta)_j}&=\sum_{j=0}^{\nu}\binom{\nu}{j}(-)^j\frac{(\alpha)_j}{(\beta)_j}+\sum_{j=1}^{\nu+1}\binom{\nu}{j-1}(-)^j\frac{(\alpha)_j}{(\beta)_j}\\&=\sum_{j=0}^{\nu}\binom{\nu}{j}(-)^j\frac{(\alpha)_j}{(\beta)_j}+\sum_{j=0}^{\nu}\binom{\nu}{j}(-)^{j+1}\frac{(\alpha)_{j+1}}{(\beta)_{j+1}}.\end{aligned}\tag{3.3}$$

Since $(a)_{j+1}=a\cdot(a+1)_j$, the assumption of the induction implies that (3.3) equals

$$\frac{(\beta-\alpha)_\nu}{(\beta)_\nu}-\frac{\alpha}{\beta}\frac{(\beta-\alpha)_\nu}{(\beta+1)_\nu}=\frac{(\beta-\alpha)_{\nu+1}}{(\beta)_{\nu+1}}.$$

This completes the proof. ∎

Remarks. 1. Lemma 3.1 is also obtained by specifying the parameters in the Gauss formula

$$F(a,b;c;1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\qquad(\operatorname{Re} c>0,\ \operatorname{Re}(c-a-b)>0)$$

as $a=-\nu$, $b=\alpha$ and $c=\beta$, where $F(a,b;c;z)$ is the hypergeometric function given by

$$F(a,b;c;z)=\sum_{j=0}^{\infty}\frac{(a)_j\,(b)_j}{(c)_j}\frac{z^n}{n!}.$$

(See, e.g., [WW].)

2. Since $(\beta)_\nu/(\beta)_j=(\beta+j)_{\nu-j}$ for $0\leqslant j\leqslant \nu$, we can rewrite (3.2) as

$$\sum_{j=0}^{\nu}\binom{\nu}{j}(-)^j(\alpha)_j(\beta+j)_{\nu-j}=(\beta-\alpha)_\nu,\tag{3.4}$$

which also holds if $\beta=0$.

LEMMA 3.2. *Let ν and i be integers such that $0\leqslant i\leqslant \nu-1$. Then for $\alpha,\beta\in\mathbb{C}\backslash\mathbb{Z}$, we have*

$$\sum_{j=0}^{i} \binom{v}{j} (-)^{i-j} (\alpha+i-j)_{j+1} (1-\beta)_{v-i} (\beta)_{i-j+1}$$
$$+ \sum_{j=i+1}^{v} \binom{v}{j} (t)^{i-j+1} (-\beta+j-i)_{v-j+1} (\alpha)_{i+1} (1-\alpha)_{j-i}$$
$$= -(\alpha+i)(v-i-\beta)(\alpha-\beta-1)_v. \tag{3.5}$$

Proof. Using (3.1), we obtain

$$(\alpha+i-j)_{j+1} = \frac{(\alpha)_{i+1}}{(\alpha)_{i-j}} = (-)^{i-j} (\alpha)_{i+1} (1-\alpha)_{j-i},$$
$$(\beta)_{i-j+1} = \frac{(-)^{i-j+1}}{(1-\beta)_{j-i-1}}.$$

Thus the first sum of the left-hand side of (3.5) equals

$$(\alpha)_{i+1} \sum_{j=0}^{i} \binom{v}{j} (-)^{i-j+1} \frac{(1-\alpha)_{j-i} (1-\beta)_{v-i}}{(1-\beta)_{j-i-1}}.$$

Similarly, the second sum of the left-hand side of (3.5) equals

$$(\alpha)_{i+1} \sum_{j=i+1}^{v} \binom{v}{j} (-)^{i-j+1} \frac{(1-\alpha)_{j-i} (1-\beta)_{v-i}}{(1-\beta)_{j-i-1}}.$$

Since

$$(1-\alpha)_{j-i} = (1-\alpha-i)_j (1-\alpha)_{-i} = (-)^i \frac{(1-\alpha-i)_j}{(\alpha)_i},$$

we obtain, by Lemma 3.1, that the left-hand side of (3.5) equals

$$\frac{(\alpha)_{i+1}}{(\alpha)_i} (1-\beta)_{v-i} (\beta)_{i+1} (-)^i \sum_{j=0}^{v} \binom{v}{j} (-)^j \frac{(1-\alpha-i)_j}{(1-\beta-i-1)_j}$$
$$= (\alpha+i)(1-\beta)_{v-i} (\beta)_{i+1} (-)^i \frac{(\alpha-\beta-1)_v}{(-\beta-i)_v}$$
$$= -(\alpha+i)(v-i-\beta)(\alpha-\beta-1)_v.$$

This completes the proof. ∎

Remark. We can show that Lemma 3.2 also holds if $\alpha=0$ or $\beta=1$ in the same manner as above. Notice that if $\alpha=0$ then the second sum of the right-hand side in (3.5) vanishes, and if $\beta=0$ the first sum vanishes.

PROPOSITION 3.1. *Let p be a nonnegative integer and z a complex number. Then we have*

$$e^{-zL_p} L_q e^{zL_p} u = \exp(-z \operatorname{ad} L_p)\, L_q u \tag{3.6}$$

for any element u of $\mathscr{U}$, where $q = \pm 1$.

Proof. We shall only prove the case where $q = -1$, since the case where $q = 1$ can be proved similarly. We may assume that u is of the form $\exp \langle \mathbf{c}, \mathbf{x} \rangle$ as usual. Then it follows from (1.9) and (1.10) that the left-hand side of (3.6) equals

$$(-\eta_p(d_{-1}(D_p(\mathbf{c}, z)), D_p(\mathbf{c}, z); -z) + \langle D_p(d_{-1}(D_p(\mathbf{c}, z)), -z), \mathbf{x} \rangle)\, u. \tag{3.7}$$

The coefficient of $z^\nu/\nu!$, $\nu \geqslant 0$, in $\langle D_p(d_{-1}(D_p(\mathbf{c}, z)), -z), \mathbf{x} \rangle$ equals

$$\sum_{m \geqslant 1} \sum_{k=0}^{\nu} \binom{\nu}{k} (-)^k\, m(m+p) \cdots (m+kp)$$
$$\times (m-1+kp)(m-1+(k+1)\,p) \cdots (m-1+\nu p)\, c_{m-1+\nu} x_m.$$

Then, it follows from (3.4) that

$$\begin{aligned}
&\sum_{k=0}^{\nu} \binom{\nu}{k} (-)^k\, (m+p) \cdots (m+kp) \\
&\qquad \times (m-1+kp)(m-1+(k+1)\,p) \cdots (m-1+\nu p) \\
&= p^\nu \sum_{k=0}^{\nu} \binom{\nu}{k} (-)^k \left(\frac{m}{p}+1\right) \cdots \left(\frac{m}{p}+k\right)\left(\frac{m-1}{p}+k\right) \cdots \\
&\qquad \times \left(\frac{m-1}{p}+(\nu-1)\right) \\
&= p^\nu \left(\frac{m-1}{p} - \frac{m}{p} - 1\right)_\nu = (-1-p)_\nu.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\langle D_p(d_{-1}(D_p(\mathbf{c}, z)), -z), \mathbf{x} \rangle \\
&\qquad = \sum_{\nu \geqslant 0} \frac{z^\nu}{\nu!} (-1-p)_\nu \sum_{m \geqslant 1} m(m-1+\nu p)\, c_{m-1+\nu p} x_m.
\end{aligned} \tag{3.8}$$

Now we turn to the polarized quadratic term in (3.7). From the definition (1.15) and the identity

$$\sum_{l=0}^{\mu} \binom{\mu}{l} \frac{(-)^l}{k+l+1} = \frac{k!\,\mu!}{(k+\mu+1)!}$$

for $k, \mu \in \mathbb{Z}_{\geqslant 0}$, it follows that

$$\begin{aligned}
&\eta_p(d_{-1}(D_p(\mathbf{c}, z)), D_p(\mathbf{c}, z); -z) \\
&\quad = -z \sum_{m=1}^{p-1} \sum_{k, \mu \geqslant 0} \frac{(-z)^k z^\mu}{k!\,\mu!} m(m+p) \cdots \\
&\qquad \times (m+kp)(m-1+kp)\, D_p(\mathbf{c}, z)_{m-1+kp} \\
&\qquad \times (p-m)(p-m+p) \cdots (p-m+\mu p)\, c_{p-m+\mu p} \sum_{l=0}^{\mu} \binom{\mu}{l} \frac{(-)^l}{l+l+1} \\
&\quad = -z \sum_{m=1}^{p-1} \sum_{k, l, j \geqslant 0} \frac{(-)^k z^{k+l+j}}{(k+l+1)!} m(m+p) \cdots (m+kp) \\
&\qquad \times (m-1+kp)(m-1+(k+1)\, p) \cdots \\
&\qquad \times (m-1+(k+j)\, p)\, c_{m-1+(k+j)\, p} \\
&\qquad \times (p-m)(p-m+p) \cdots (p-m+\mu p)\, c_{p-m+lp}.
\end{aligned} \tag{3.9}$$

Thus the coefficient of $z^\nu/\nu!$, $\nu \geqslant 1$, in (3.9) equals

$$\begin{aligned}
&-\sum_{m=1}^{p-1} \sum_{\substack{k, l, j \geqslant 0 \\ k+l+j+1=\nu}} \frac{\nu!}{(k+l+1)!\, j!} (-)^k m(m+p) \cdots (m+kp) \\
&\quad \times (m-1+kp) \cdots (m-1+(k+j)\, p)\, c_{m-1+(k+j)\, p} \\
&\quad \times (p-m) \cdots (p-m+lp)\, c_{p-m+lp}.
\end{aligned} \tag{3.10}$$

Now it is easy to show that (3.10) equals

$$\begin{aligned}
&-\frac{1}{2} p^{\nu+2} \sum_{m=0}^{p-1} \sum_{i=0}^{\nu-1} c_{m+ip} c_{(\nu-i)\, p-1-m} \\
&\quad \times \left\{ \sum_{k=0}^{i} \binom{\nu}{i-k} (-)^k \left(\frac{m}{p}+k\right)_{i-k+1} \left(1-\frac{m+1}{p}\right)_{\nu-i} \left(\frac{m+1}{p}\right)_{k+1} \right. \\
&\quad + \sum_{k=0}^{\nu-1-i} \binom{\nu}{i+k+1} (-)^k \left(\frac{m}{p}\right)_{i+1} \\
&\quad \left. \times \left(1-\frac{m+1}{p}+k\right)_{\nu-i-k} \left(1-\frac{m}{p}\right)_{k+1} \right\}.
\end{aligned}$$

Applying Lemma 3.2 and Remark that follows the lemma with $\alpha = m/p$, $\beta = (m+1)/p$, we obtain that the coefficient of $c_{m+ip} c_{(v-i)\,p-1-m}$ equals

$$-p^{v+2}\left(\frac{m}{p}+i\right)\left((v-i)-\frac{m+1}{p}\right)\left(-\frac{1}{p}-1\right)_v \tag{3.11}$$

for $i = 0, 1, ..., v-1$, $m = 0, 1, ..., p-1$.

Therefore, (3.11), together with (3.8), implies that the coefficient of $z^v/v!$, $v \geqslant 1$, in (3.7) equals

$$p^v\left(-\frac{1}{p}-1\right)_v \left(-\frac{1}{2}\sum_{m=1}^{-1+vp-1} m(-1+vp-m)\, c_m c_{-1+vp-m} \right.$$
$$\left. + \sum_{m \geqslant 1} m(m+vp-1)\, x_m c_{m+vp-1}\right) u$$

which is nothing but that of $z^v/v!$ in the right-hand side of (3.6). The case where $v = 0$ is trivial. This completes the proof. ∎

For $k \geqslant 1$, let $\varepsilon(k)$ denote the element of E whose nth component is given by $\delta_{k,n}$. Then we can rewrite (1.13) as

$$L_{-p}u = -\frac{1}{2}\sum_{n=1}^{p-1} \frac{\partial^2}{\partial s_1\,\partial s_2}\bigg|_{s_1=s_2=0} \exp\langle \mathbf{c} + s_1\varepsilon(n) + s_2\varepsilon(p-n), \mathbf{x}\rangle$$
$$+ \frac{\partial}{\partial s}\bigg|_{s=0} \exp\langle \mathbf{c} + s d_{-p}(\mathbf{c}), \mathbf{x}\rangle \tag{3.12}$$

for $p \geqslant 0$ and $u = \exp\langle \mathbf{c}, \mathbf{x}\rangle$.

Proposition 3.2. *Under the same assumption as in Proposition* 3.1, *we have*

$$e^{-zL_p} L_q e^{zL_p} u = \exp(-z \operatorname{ad} L_p)\, L_q u \tag{3.13}$$

for any element $u \in \mathscr{U}$, where $q = \pm p$.

Proof. If $q = p$ then the left-hand side can be written as

$$e^{-zL_p}\left(\frac{d}{dz_1}\bigg|_{z_1=0} e^{z_1 L_p}\right) e^{z_1 L_p} u = \frac{d}{dz_1}\bigg|_{z_1=0} e^{z_1 L_p} u$$
$$= L_p u,$$

which shows that the proposition is true in this case, for $(\operatorname{ad} L_p)^v L_p = 0$ if $v \geqslant 1$.

Now the case where $q=-p$. Notice that, in the right-hand side of (3.13), the sum $\sum_{\nu\geqslant 3}$ vanishes since $(\operatorname{ad} L_p)^3 L_{-p}=0$. It follows from (1.19), (1.10) and (3.12) that the left-hand side of (3.13) equals

$$\begin{aligned}
-\tfrac{1}{2}\sum_{n=1}^{p-1} & (-\eta_p(\varepsilon(n), \varepsilon(p-n); -z) \\
& + \eta_p(\varepsilon(n), D_p(\mathbf{c}, z); -z)\, \eta_p(\varepsilon(p-n), D_p(\mathbf{c}, z); -z) \\
& - 2\eta_p(\varepsilon(n), D_p(\mathbf{c}, z); -z) \langle D_p(\varepsilon(p-n), -z), \mathbf{x}\rangle \\
& + \langle D_p(\varepsilon(n), -z), \mathbf{x}\rangle \langle D_p(\varepsilon(p-n), -z), \mathbf{x}\rangle)\, u \\
& + (-\eta_p(d_{-p}(D_p(\mathbf{c}, z)), D_p(\mathbf{c}, z); -z) \\
& + \langle D_p(d_{-p}(D_p(\mathbf{c}, z)), -z), \mathbf{x}\rangle)\, u. \qquad (3.14)
\end{aligned}$$

Now we calculate (3.14) term by term. First, it is easy to show that

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{p-1} & \eta_p(\varepsilon(n), \varepsilon(p-n); -z) \\
&= -\frac{z}{2}\sum_{n=1}^{p-1} n(p-n), \\
&\quad -\frac{1}{2}\sum_{n=1}^{p-1} \langle D_p(\varepsilon(n), -z), \mathbf{x}\rangle \langle D_p(\varepsilon(p-n), -z), \mathbf{x}\rangle \\
&= -\frac{1}{2}\sum_{n=1}^{p-1} n(p-n)\, x_n x_{p-n}. \qquad (3.15)
\end{aligned}$$

Next, since

$$\eta_p(\varepsilon(n), D_p(\mathbf{c}, z); -z) = -\sum_{k\geqslant 1} \frac{z^k}{k!} n(p-n)\cdots(kp-n)\, c_{kp-n},$$

we obtain that the coefficient of $z^\nu/\nu!$, $\nu\geqslant 2$, in

$$-\frac{1}{2}\sum_{n=1}^{p-1} \eta_p(\varepsilon(n), D_p(\mathbf{c}, z); -z)\, \eta_p(\varepsilon(p-n), D_p(\mathbf{c}, z); -z) \qquad (3.16)$$

equals

$$\begin{aligned}
-\tfrac{1}{2}\sum_{n=1}^{p-1}\sum_{k=1}^{\nu} & \binom{\nu}{k} n(p-n)\cdots(kp-n)\, c_{kp-n} \\
& \times (p-n)\, n(n+p)\cdots(n+(\nu-k-1)\, p)\, c_{n+(\nu-k-1)\, p}.
\end{aligned}$$

Similarly, simple calculation shows that the coefficient of $z^\nu/\nu!$, $\nu \geqslant 2$, in

$$\begin{aligned}
&-\eta_p(d_{-p}(D_p(\mathbf{c}, z), D_p(\mathbf{c}, z); -z)) \\
&\quad = \sum_{n=1}^{p-1} \sum_{k,\, l \geqslant 0} \frac{z}{k+l+1} \frac{(-z)^k}{k!} n(n+p) \cdots (n+kp)\, d_{-p}(D_p(\mathbf{c}, z))_{n+kp} \\
&\qquad \times \frac{z^k}{l!} (p-n) \cdots (p-n+lp)\, D_p(\mathbf{c}, z)_{p-n+lp} \qquad (3.17)
\end{aligned}$$

equals

$$\begin{aligned}
&-\frac{1}{2} \sum_{n=1}^{p-1} \sum_{\substack{i,\, l \geqslant 1 \\ i+l=\nu}} \binom{\nu}{i} n(n+p) \cdots (n+(i-1)\,p)\, c_{n+(i-1)\,p} \\
&\qquad \times (p-n) \cdots (lp-n)\, c_{lp-n} \\
&\qquad \times \left(\sum_{k=1}^{i} \binom{i}{k} (-)^k\, (n+(k-1)\,p)\, (n+kp)\, \frac{k!\; l!}{(k+l)!} \right. \\
&\qquad \left. + \sum_{k=1}^{l} \binom{l}{k} (-)^k\, (kp-n)((k+1)\,p-n)\, \frac{k!\; i!}{(k+i)!} \right).
\end{aligned}$$

Using the identity

$$\begin{aligned}
&\sum_{k=1}^{i} \binom{i}{k} (-)^k\, (n+(k-1)\,p)(n+kp)\, \frac{k!(\nu-i)!}{(k+\nu-i)!} \\
&\qquad + \sum_{k=1}^{\nu-i} \binom{l}{k} (-)^k\, (kp-n)((k+1)\,p-n)\, \frac{k!\; i!}{(k+i)!} \\
&\quad = \binom{\nu}{i}^{-1} \sum_{\substack{0 \leqslant j \leqslant \nu \\ j \neq i}} \binom{\nu}{j} (-)^{i-j}\, (n+(i-j)\,p)(n+(i-j-1)\,p),
\end{aligned}$$

we obtain that the coefficient of $z^\nu/\nu!$, $\nu \geqslant 1$, in the sum of (3.16) and (3.17) equals

$$\tfrac{1}{2} \sum_{n=1}^{p-1} n(p-n)\, c_n c_{p-n} \times (-2p^2)\, \delta_{\nu,\, 2}. \qquad (3.18)$$

Finally, the coefficient of $z^\nu/\nu!$, $\nu \geqslant 1$, in

$$\begin{aligned}
&\eta_p(\varepsilon(n), D_p(\mathbf{c}, z); -z) \langle D_p(\varepsilon(p-n), -z), \mathbf{x} \rangle \\
&\quad = -\sum_{k \geqslant 1} \frac{z^k}{k!} (p-n)\, n(n+p) \cdots (n+(k-1)\,p)\, c_{n+(k-1)\,p} \cdot n x_n, \qquad (3.19)
\end{aligned}$$

equals

$$-\sum_{n=1}^{p-1} (p-n)\, n(n+p) \cdots (n+(\nu-1)\, p)\, c_{n+(\nu-1)\, p} x_n \cdot n(p-n).$$

Furthermore, the coefficient of $z^\nu/\nu!$, $\nu \geqslant 0$, in

$$\begin{aligned}
&\langle D_p(d_{-p}(D_p(\mathbf{c}, z)), -z), \mathbf{x} \rangle \\
&\quad = \sum_{n \geqslant 1} n x_n \sum_{k,\, j \geqslant 0} \frac{(-z)^k\, z^j}{k!\, j!} (n+p) \cdots (n+kp)(n+kp-p)\, \theta(n+kp-p) \\
&\qquad \times (n+kp)(n+(k+1)\, p) \cdots (n+(k+j-1)\, p)\, c_{n+(k+j-1)\, p} \qquad (3.20)
\end{aligned}$$

equals

$$\begin{aligned}
&\sum_{n \geqslant p+1} n x_n (n-p) \cdot n(n+p) \cdots (n+(\nu-1)\, p)\, c_{n+(\nu-1)\, p} \\
&\quad + \sum_{n \geqslant 1} n x_n (n+p) \cdots (n+(\nu-1)\, p)\, c_{n+(\nu-1)\, p} \\
&\quad \times \sum_{k=1}^{\nu} \binom{\nu}{k} (-)^k\, (n+(k-1)\, p)(n+kp).
\end{aligned}$$

Therefore, we obtain that, for $\nu \geqslant 0$, the coefficient of $z^\nu/\nu!$ in the sum of (3.19) and (3.20) equals

$$\sum_{n \geqslant 1} n(n+p)\, c_n x_{n+p} \qquad (3.21)$$

if $\nu = 0$, and equals

$$\begin{aligned}
&\sum_{n \geqslant 1} n(n+p) \cdots (n+(\nu-1)\, p)\, c_{n+(\nu-1)\, p} x_n \\
&\qquad \times \sum_{k=0}^{\nu} \binom{\nu}{k} (-)^k\, (n+(k-1)\, p)(n+kp)
\end{aligned}$$

$$= \begin{cases} -2p \sum_{n \geqslant 1} n^2 x_n c_n & \text{if} \quad \nu = 1, \\ 2p^2 \sum_{n \geqslant 1} n(n+p)\, x_n c_{n+p} & \text{if} \quad \nu = 2, \\ 0 & \text{if} \quad \nu \geqslant 3. \end{cases} \qquad (3.22)$$

Now it follows immediately from (3.15), (3.18), (3.21) and (3.22) that the coefficient of $z^\nu/\nu!$ in the left-hand of (3.13) is equal to that of $z^\nu/\nu!$ in the right-hand side of (3.13) for each $\nu \geqslant 0$. This completes the proof. ∎

THEOREM 3.1. *Let p be an integer and z a complex number. For an element X of Vir, we have*

$$e^{-zL_p}\pi(X)\, E^{zL_p} = \exp(-z \operatorname{ad} L_p)\, \pi(X).$$

Proof. If X is in the center of Vir, then it is trivial. Using the fact that ad is a derivation, we obtain that

$$\exp(-z \operatorname{ad} X)([Y, Z]) = [\exp(-z \operatorname{ad} X)\, Y, \exp(-z \operatorname{ad} X) Z].$$

Thus, it follows from Propositions 3.1, 3.2 and the commutation relation (1.1) that, for $p \geqslant 0$ and $u \in \mathscr{U}$,

$$\begin{aligned}
e^{-zL_p} L_{p\pm 1} e^{zL_p} u &= \frac{1}{p \mp 1} [e^{-zL_p} L_p e^{zL_p}, e^{-zL_p} L_{\pm 1} e^{zL_p}]\, u \\
&= \frac{1}{p \mp 1} [\exp(-z \operatorname{ad} L_p)(L_p), \exp(-z \operatorname{ad} L_p)(L_{\pm 1})]\, u \\
&= \frac{1}{p \mp 1} \exp(-z \operatorname{ad} L_p)([L_p, L_{\pm 1}])\, u \\
&= \exp(-z \operatorname{ad} L_p)\, L_{p\pm 1} u.
\end{aligned}$$

Repeating the argument above, we obtain that

$$e^{-zL_p} L_q e^{zL_p} u = \exp(-z \operatorname{ad} L_p)\, L_q u \tag{3.23}$$

for $q > 0$. The case where $q \leqslant 0$ follows similarly.

Now let us assume that $p < 0$. Then, from Proposition 1.1 and the definition of the composition of the operators, it follows that

$$\begin{aligned}
&\langle u|\, e^{-zL_p} L_q e^{zL_p}\, |v\rangle \\
&\quad = \iint d\mu(\mathbf{z}_1)\, d\mu(\mathbf{z}_2) \langle u|\, e^{-zL_p}\, |e_{\mathbf{z}_1}\rangle \langle e_{\mathbf{z}_1}|\, L_q\, |e_{\mathbf{z}_2}\rangle \langle e_{\mathbf{z}_2}|\, e^{zL_p}\, |v\rangle \\
&\quad = \overline{\langle v|\, e^{\bar{z}L_{-p}} L_{-q} e^{-\bar{z}L_{-p}} u\rangle},
\end{aligned}$$

which, by (3.23), equals

$$\begin{aligned}\overline{\langle v|\exp(\bar{z}\,\mathrm{ad}\,L_{-p})\,L_{-q}u\rangle} &= \sum_{n\geqslant 0}\frac{\bar{z}^n}{n!}\overline{\langle v|(\mathrm{ad}\,L_{-p})^n\,L_{-q}u\rangle}\\ &= \sum_{n\geqslant 0}\frac{\bar{z}^n}{n!}\langle u|(-\mathrm{ad}\,L_p)^n\,L_q v\rangle\\ &= \langle u|\exp(-z\,\mathrm{ad}\,L_p)\,L_q\,|v\rangle\end{aligned}$$

for all $u, v \in \mathscr{U}$, where we used the identity that

$$\langle v|(\mathrm{ad}\,L_{-p})^n\,L_{-q}u\rangle = \langle(-\mathrm{ad}\,L_p)^n\,L_q v|u\rangle$$

for $n = 0, 1, 2, ...$ This completes the proof of the theorem. ∎

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